Time: 75 minutes **Calculator: OK**
Notes: 8$\frac{1}{2}$"x11"sheet front and back (or equivalent)

Directions: For multiple choice (A, B, C, etc.), **circle** *the letters* of all correct choices. For TRUE/FALSE, **circle** *the word* TRUE or *the word* FALSE. Otherwise follow the directions given.

Unless told otherwise, **all** numbers in answers should be as ***common fractions*** in *reduced* form or in radical form similarly reduced. (You may also use decimal form if the value is terminating or proper notation is used to denote the repeating digit sequence.)

Work easier problems first. Write out your plan for harder problems before beginning them.
Note that the last two pages are bonus!

1) Construct a truth table for this compound proposition: $(p \to q) \wedge (\neg p \to q)$.

p	q	$\frac{A}{p\to q}$	$\neg p$	$\frac{B}{\neg p \to q}$	$A \wedge B$
true	**true**	true	false	true	**true**
true	**false**	false	false	true	**false**
false	**true**	true	true	true	**true**
false	**false**	true	true	false	**false**

What is the proposition equivalent to?

A) p B) q C) $p \wedge q$ D) $p \vee q$

2) Use De Morgan's laws to find the negation of each of the following statements.

i) Jan is rich and happy.
Jan is poor or unhappy.

ii) Carlos will bicycle or run tomorrow.
Carlos will not bicycle and will not run tomorrow.

3) Determine the truth value of each of the following statements if the domain for all variables consists of all integers.

i) $\forall n \exists m(n + m = 0)$
All integers have an additive inverse, thus this statement is **true**.

ii) $\exists n \exists m(n + m = 4 \wedge n - m = 1)$
If $n + m = 4$, then $n = 4 - m$. Thus $n - m = 4 - m - m = 4 - 2m$. But $4 - 2m$ must be even for any integer m, thus this statement is **false**.

iii) $\exists n \exists m(n^2 + m^2 = 5)$
If we let $n = 1$ and $m = 2$, then $n^2 + m^2 = 1 + 4 = 5$. Thus this statement is **true**.

iv) $\exists n \forall m(n < m^2)$
The square of all integers is zero or more, thus this statement is **true**, if we simply choose n to be any negative.

4) Prove that there is a positive integer that equals the sum of the positive integers not exceeding it. Is your proof constructive or nonconstructive?
One is the sum of one and doesn't exceed itself. Therefore, such an integer exists. (This proof is constructive.)

5) Evaluate the following implications.

TRUE / FALSE If $1+1=2$, then $2+2=5$.

TRUE / FALSE If $1+1=3$, then $2+2=4$.

TRUE / FALSE If $1+1=3$, then $2+2=5$.

6) Use the contrapositive approach (indirect proof) to show that for every positive real number r, if r is irrational, then $\sqrt{r}$ is also irrational.
The contrapositive would be that for every positive real number r, if $\sqrt{r}$ is rational, then so is r. Let $\sqrt{r}=\frac{x}{y}$ such that x and y are both integers and $y \neq 0$. Squaring both sides we find that:

$$\sqrt{r}^2 = \left(\frac{x}{y}\right)^2$$
$$r = \frac{x^2}{y^2}$$

Since x and y are both integers, x^2 and y^2 are also integers. Further, since $y \neq 0$, $y^2 \neq 0$. Thus, r is the ratio of two integers whose second number is not 0 making it rational. Since the contrapositive is true, so is the original implication!

7) Prove that if x and y are real numbers, then $\max(x,y)+\min(x,y)=x+y$. [Hint: Use a proof by cases, with the two cases corresponding to $x \geqslant y$ and $x<y$, respectively.]
Let's take first the case where $x \geqslant y$. In this situation, $\max(x,y)=x$ and $\min(x,y)=y$. Thus, $\max(x,y)+\min(x,y)=x+y$, and we are done.
In the second case where $x<y$, $\max(x,y)=y$ and $\min(x,y)=x$. Thus, $\max(x,y)+\min(x,y)=y+x=x+y$, and we are done.
Since it is true in both cases and the cases considered cover all possibilities, it must be true in general.

8) Use a proof by contradiction to prove that for $a, b,$ and c integers with $a \neq 0$ that if $a \nmid bc$, then $a \nmid b$.
Suppose that $a|b$. Then a must also divide bc. But this contradicts our premise that $a \nmid bc$! Therefore a must not actually divide b.

9) Use a direct proof to show that the sum of two odd integers is even.
If m is odd, then it is $2k+1$ for some integer k. Therefore, the sum of **two** odd integers would be $(2k+1)+(2j+1)=2k+2j+2=2(k+j+1)$. Since this result is clearly even, we have our proof.

10) You are given this fact: "If $n>2$ is a prime, then n is odd." Then you are asked to prove this proposition: "If p is a prime with $p>2$, then $p+1$ is not prime." [Note that this proposition is *for all* primes greater than 2.]
Using the choose method, we pick a representative prime from the set involved — those greater than 2 — call it k. This prime must be odd by the original fact given. Since k is odd, it must be representable as $2m+1$ for some integer m. If we add one to k, that must be equal to $2m+1+1=2m+2=2(m+1)$. Since this is clearly even, it is divisible by 2 — but not 2 — and so is not prime. □

Bonus Problems:

11) State what the following statement means when the *or* is inclusive vs. when it is exclusive. Which type of *or* do you think was intended?

Dinner for two includes two items from column A or three items from column B.

With an inclusive *or*, the statement means that either two column-A items or three column-B items or both are included with the two-person dinner.

With an exclusive *or*, the statement means that either two column-A items or three column-B items but not both are included with the two-person dinner.

I believe they meant exclusive *or* here. The restaurant won't get rich selling five items for the same price as two or three.

12) Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers.

i) $\exists x \forall y (xy = y)$

There is a multiplicative identity for the real numbers such that any real number multiplied by this quantity produces the original number identically.

ii) $\forall x \forall y \exists z (x + y = z)$

The real numbers are closed under addition. That is, adding any two real numbers results in a third real number.

13) Show that $(p \to q) \to r$ and $p \to (q \to r)$ are not equivalent. (And thus that implication is not associative.)

This can be done with a truth table:

p	q	r	A: $p \to q$	B: $A \to r$	C: $q \to r$	D: $p \to C$
true	true	true	true	true	true	true
true	true	false	true	false	false	false
true	false	true	false	true	true	true
true	false	false	false	true	true	true
false	true	true	true	true	true	true
false	true	false	true	**false**	false	**true**
false	false	true	true	true	true	true
false	false	false	true	**false**	true	**true**

Note how the last and second-to-last entries differ for the propositions under consideration (columns B and D).

14) Prove the given fact from #10: "If $n > 2$ is prime, then n is odd." [Hint: The premise could have also been written as "$n > 2$ and n is prime".] [Hint 2: An indirect approach might be easiest.]

An indirect approach will require the contrapositive implication: "If n is even, then either $n \leqslant 2$ or n is not prime." If n is even, then we can write it as $2k$ for some integer k. Let's proceed by cases on k: $k \leqslant 1$ and $k > 1$.

Case 1: If $k \leqslant 1$, then $2k \leqslant 2$ and thus the left clause is true.

Case 2: If $k > 1$, then $2k > 2$. But does this mean n is not prime? Yes, because it is not 2 and yet has a factor of 2 — therefore a factor not itself or 1.

Having proved both clauses, we can use the principle of addition to combine them and, since the original implication is logically equivalent to the contrapositive, we've proven the entire 'fact'. □

15) Prove that if m and n are integers and mn is even, then either m is even or n is even.

If mn is even, then it is $2k$ for some integer k. But if $mn = 2k$, 2 must be a factor of either m or n. If 2 is a factor of m, then it is even. If 2 is a factor of n, then it is even. Therefore, either m or n must be even.

16) Express the following English sentence using predicates, quantifiers, and logical operations.

> Each participant on the conference call whom the host of the call did not put on a special list was billed.

Domain: all callers.

$C(x) = x$ participated in the conference call

$L(x) = x$ was listed specially by the conference call's host

$B(x) = x$ was billed for the conference call

$\forall x((C(x) \wedge \neg L(x)) \rightarrow B(x))$

What is the opposite of the sentence? (It may help to negate your symbolic version first.)

The opposite of $\forall x((C(x) \wedge \neg L(x)) \rightarrow B(x))$ would be:

$$\begin{aligned} \neg(\forall x((C(x) \wedge \neg L(x)) \rightarrow B(x))) &= \exists x \neg((C(x) \wedge \neg L(x)) \rightarrow B(x)) \\ &= \exists x \neg(\neg(C(x) \wedge \neg L(x)) \vee B(x)) \\ &= \exists x(C(x) \wedge \neg L(x) \wedge \neg B(x)) \end{aligned}$$

Therefore we have in English something like, "There exists a caller who was not billed for the conference call and who both did participate in the conference call and was not listed specially by the conference call's host." Since that is a bit wordy, we may wish to try harder to compact it. Perhaps, "Someone wasn't billed for the conference call and yet they were in on the call and were not given special permission by the call's host."