Directions: For multiple choice (A, B, C, etc.), **circle** *the letters* of all correct choices. For TRUE/FALSE, **circle** *the word* TRUE or *the word* FALSE. Otherwise follow the directions given.

Unless told otherwise, **all** numbers in answers should be as ***common fractions*** in *reduced* form or in radical form similarly reduced. (You may also use decimal form if the value is terminating or proper notation is used to denote the repeating digit sequence.)

Work easier problems first. Write out your plan for harder problems before beginning them. Note that the last two pages are bonus!

1) In set theory, the symmetric difference operation can be used to find the items unique to each of a pair of sets:

$$A \Delta B = (A - B) \cup (B - A)$$

This operation can be used in boolean algebra to help solve equations when the difference is 0:

$$x \Delta y = x \cdot \overline{y} + \overline{x} \cdot y$$

Show what the symmetric difference operation looks like in propositional logic.

Working from the definitions of symmetric difference in set theory and boolean algebra, derive an equivalence for the ***set*** difference operation that involves only the three basic set operations: intersection, union, and/or complement. (You do **not** need to prove your form's equivalence — just justify it.)

What might symmetric difference be used for in propositional logic?

2) Evaluate the following implications.

TRUE / FALSE If $1 + 1 = 2$, then $2 + 2 = 5$.

TRUE / FALSE If $1 + 1 = 3$, then $2 + 2 = 4$.

TRUE / FALSE If $1 + 1 = 3$, then $2 + 2 = 5$.

3) Construct a truth table for this compound proposition: $(p \to q) \land (\neg p \to q)$.

What is the proposition equivalent to?

A) p B) q C) $p \land q$ D) $p \lor q$

4) Use De Morgan's laws to find the negation of each of the following statements.

i) Jan is rich and happy.

ii) Carlos will bicycle or run tomorrow.

5) Determine the truth value of each of the following statements if the domain for all variables consists of all integers.

i) $\forall n \exists m(n + m = 0)$

ii) $\exists n \exists m(n + m = 4 \land n - m = 1)$

iii) $\exists n \exists m(n^2 + m^2 = 5)$

iv) $\exists n \forall m(n < m^2)$

6) Use the contrapositive approach (indirect proof) to show that for sets A and B, if $x \in (B - A)$, then $x \notin (A \cap B)$.

7) Find the inverse of 46, $\mod 7$. (Or state why it cannot exist.)

Find the inverse of 62, $\mod 4$. (Or state why it cannot exist.)

8) Prove that there is a positive integer that equals the sum of the positive integers not exceeding it. Is your proof constructive or nonconstructive?

9) Prove that if x and y are real numbers, then $\max(x, y) + \min(x, y) = x + y$. [Hint: Use a proof by cases, with the two cases corresponding to $x \geqslant y$ and $x < y$, respectively.]

10) Use a Venn diagram to illustrate the set of all months of the year whose names do not contain the letter **R** in the set of all months of the year.

11) Determine whether each of these statements is true or false.

   i) $0 \in \varnothing$

   ii) $\{0\} \subset \varnothing$

   iii) $\varnothing \in \{0\}$

12) Use a proof by contradiction to prove that for $a, b,$ and $c \in \mathbb{Z}$ with $a \neq 0$ that if $a \nmid bc$, then $a \nmid b$.

13) Use a direct proof to show that the sum of two odd integers is even.

14) Prove this identity law for sets: $A \cap U = A$. [Hint: What does it mean for two sets to be equal?]

15) Letting $A = \{a, b, d\}$ and $B = \{a, b, c, e, f\}$, find the following:

i) $A \cap B$ ii) $A \cup B$ iii) $A - B$ iv) $B - A$

16) Prove by induction:

$$\sum_{k=0}^{n} \left(\frac{-1}{2}\right)^k = \frac{2}{3} + \frac{1}{3} \cdot \left(\frac{-1}{2}\right)^n$$

for all natural numbers n. (That is, $n \geqslant 0$.)

17) You are given this fact: "If $n > 2$ is a prime, then n is odd." Then you are asked to prove this proposition: "If p is a prime with $p > 2$, then $p + 1$ is not prime." [Note that this proposition is *for all* primes greater than 2.]

Bonus Problems:

18) State what the following statement means when the *or* is inclusive vs. when it is exclusive. Which type of *or* do you think was intended?

    Dinner for two includes two items from column A or three items from column B.

19) Verify this distributive law of Boolean algebra: $x \cdot (y + z) \equiv (x \cdot y) + (x \cdot z)$.

    Now verify this distributive law (also of Boolean algebra): $x + (y \cdot z) \equiv (x + y) \cdot (x + z)$.

20) Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers.
    i) $\exists x \forall y (xy = y)$

    ii) $\forall x \forall y \exists z (x + y = z)$

21) The term "relatively prime" means ____.
    A) a number has only one or two more factors beyond itself and 1
    B) a pair of values share no common factors other than 1
    C) he was my first cousin until the divorce
    D) one of the pair of values must be prime
    E) primality is a relative thing and must be gauged in context

22) The Well-Ordering Principle only applies ____.
    A) in conjunction with an existential quantifier
    B) to non-empty sets
    C) to sets of integers
    D) to sets of natural numbers
    E) when using induction

23) What is the power set of $\{a,b,c\}$?

24) Show that $(p \to q) \to r$ and $p \to (q \to r)$ are not equivalent. (And thus that implication is not associative.)

25) Prove the given fact from #17: "If $n > 2$ is prime, then n is odd." [Hint: The premise could have also been written as "$n > 2$ and n is prime".] [Hint 2: An indirect approach might be easiest.]

26) Express the following English sentence using predicates, quantifiers, and logical operations.

Each participant on the conference call whom the host of the call did not put on a special list was billed.

What is the opposite of the sentence? (It may help to negate your symbolic version first.)