Directions: For multiple choice (A, B, C, etc.), **circle** *the letters* of all correct choices. For TRUE/FALSE, **circle** *the word* TRUE or *the word* FALSE. Otherwise follow the directions given.

Unless told otherwise, **all** numbers in answers should be as ***common fractions*** in *reduced* form or in radical form similarly reduced. (You may also use decimal form if the value is terminating or proper notation is used to denote the repeating digit sequence.)

Work easier problems first. Write out your plan for harder problems before beginning them. Note that the last two pages are bonus!

1) In set theory, the symmetric difference operation can be used to find the items unique to each of a pair of sets:

$$A \Delta B = (A - B) \cup (B - A)$$

This operation can be used in boolean algebra to help solve equations when the difference is 0:

$$x \Delta y = x \cdot \overline{y} + \overline{x} \cdot y$$

Show what the symmetric difference operation looks like in propositional logic.
Substituting and for $\cdot$, or for $+$, and not for complement we get:

$$p \Delta q = (p \wedge \neg q) \vee (\neg p \wedge q)$$

Working from the definitions of symmetric difference in set theory and boolean algebra, derive an equivalence for the ***set*** difference operation that involves only the three basic set operations: intersection, union, and/or complement. (You do **not** need to prove your form's equivalence — just justify it.)
Since $\cup$ is equivalent to $+$ from boolean algebra, one might rightly think that set difference $(A - B)$ would be equivalent to the "$x \cdot \overline{y}$" part of the definition. But then we have to translate this back into set operations using $\cap$ for $\cdot$ and complement for, well, itself. This gives us:

$$A - B = A \cap \overline{B}$$

What might symmetric difference be used for in propositional logic?
Playing with the propositional logic form of symmetric difference, we can see that it is the logical opposite of a bi-implication:

$$\begin{aligned}
(p \wedge \neg q) \vee (\neg p \wedge q) &= (\neg q \wedge p) \vee \neg(p \vee \neg q) \\
&= \neg(q \vee \neg p) \vee \neg(q \to p) \\
&= \neg(p \to q) \vee \neg(p \to q) \\
&= \neg((p \to q) \wedge (q \to p)) \\
&= \neg(p \leftrightarrow q)
\end{aligned}$$

And, as we've seen before, exclusive or is the logical opposite of a bi-implication! So symmetric difference is the same as XOR!

2) Evaluate the following implications.

TRUE / FALSE If $1 + 1 = 2$, then $2 + 2 = 5$.

TRUE / FALSE If $1 + 1 = 3$, then $2 + 2 = 4$.

TRUE / FALSE If $1 + 1 = 3$, then $2 + 2 = 5$.

3) Construct a truth table for this compound proposition: $(p \rightarrow q) \wedge (\neg p \rightarrow q)$.

p	q	$\frac{A}{p \rightarrow q}$	$\neg p$	$\frac{B}{\neg p \rightarrow q}$	$A \wedge B$
true	**true**	true	false	true	**true**
true	**false**	false	false	true	**false**
false	**true**	true	true	true	**true**
false	**false**	true	true	false	**false**

What is the proposition equivalent to?

A) p B) q C) $p \wedge q$ D) $p \vee q$

4) Use De Morgan's laws to find the negation of each of the following statements.

i) Jan is rich and happy.
Jan is poor or unhappy.

ii) Carlos will bicycle or run tomorrow.
Carlos will not bicycle and will not run tomorrow.

5) Determine the truth value of each of the following statements if the domain for all variables consists of all integers.

i) $\forall n \exists m (n + m = 0)$
All integers have an additive inverse, thus this statement is **true**.

ii) $\exists n \exists m (n + m = 4 \wedge n - m = 1)$
If $n + m = 4$, then $n = 4 - m$. Thus $n - m = 4 - m - m = 4 - 2m$. But $4 - 2m$ must be even for any integer m, thus this statement is **false**.

iii) $\exists n \exists m (n^2 + m^2 = 5)$
If we let $n = 1$ and $m = 2$, then $n^2 + m^2 = 1 + 4 = 5$. Thus this statement is **true**.

iv) $\exists n \forall m (n < m^2)$
The square of all integers is zero or more, thus this statement is **true**, if we simply choose n to be any negative.

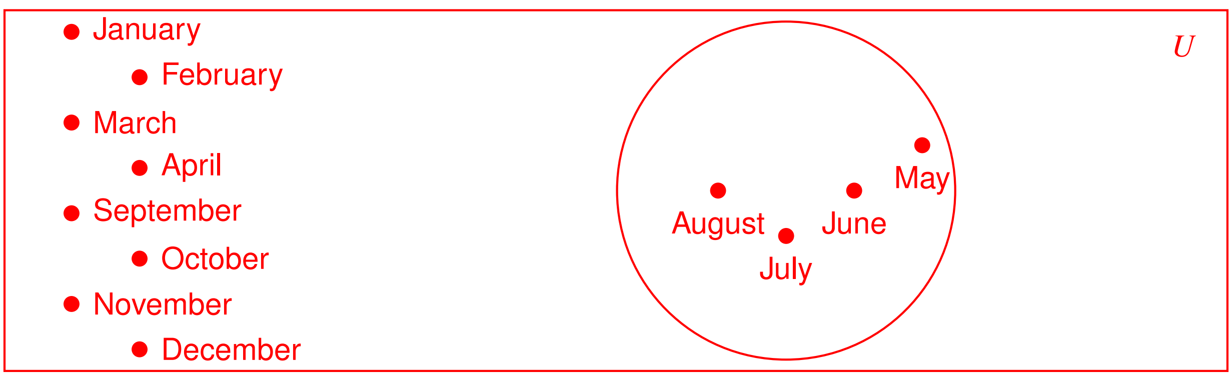
6) Use the contrapositive approach (indirect proof) to show that for sets A and B, if $x \in (B - A)$, then $x \notin (A \cap B)$.
The contrapositive would be that if $x \in (A \cap B)$, then $x \notin (B - A)$. If $x \in (A \cap B)$, then x is an element of both A and B. Since $B - A$ is determined by only those elements in B but **not** in A, x could not be in both A and $B - A$ at the same time. Therefore we have that $x \notin (B - A)$. Since the contrapositive is true, so is the original implication!

7) Find the inverse of 46, $\mod 7$. (Or state why it cannot exist.)
Since $\gcd(46, 7) = 1$, this inverse must exist. Firstly, 46 is equivalent to -3 or 4 $(\mod 7)$. The multiples of 4 $(\mod 7)$ are 4, 1... — bingo! Thus the inverse of 46 under a modulo by 7 is 2.
Find the inverse of 62, $\mod 4$. (Or state why it cannot exist.)
Since $\gcd(62, 4) = 2$, there can be no such inverse.

8) Prove that there is a positive integer that equals the sum of the positive integers not exceeding it. Is your proof constructive or nonconstructive?
One is the sum of one and doesn't exceed itself. Therefore, such an integer exists. (This proof is constructive.)

9) Prove that if x and y are real numbers, then $\max(x,y)+\min(x,y) = x+y$. [Hint: Use a proof by cases, with the two cases corresponding to $x \geqslant y$ and $x < y$, respectively.]
Let's take first the case where $x \geqslant y$. In this situation, $\max(x,y) = x$ and $\min(x,y) = y$. Thus, $\max(x,y)+\min(x,y) = x+y$, and we are done.
In the second case where $x < y$, $\max(x,y) = y$ and $\min(x,y) = x$. Thus, $\max(x,y)+\min(x,y) = y+x = x+y$, and we are done.
Since it is true in both cases and the cases considered cover all possibilities, it must be true in general.

10) Use a Venn diagram to illustrate the set of all months of the year whose names do not contain the letter **R** in the set of all months of the year.



11) Determine whether each of these statements is true or false.

i) $0 \in \varnothing$
Since there are no elements in the empty set, this must be **false**.

ii) $\{0\} \subset \varnothing$
Again, there are no elements in the empty set, therefore there can be no non-empty subsets (like this set containing the integer 0). Therefore, this is **false**.

iii) $\varnothing \in \{0\}$
The set on the right contains only the integer 0. Therefore, the empty set is not in it. (The empty set is a subset of it, but not a member of it.) Therefore, this is **false**.

12) Use a proof by contradiction to prove that for $a, b,$ and $c \in \mathbb{Z}$ with $a \neq 0$ that if $a \nmid bc$, then $a \nmid b$.
Suppose that $a|b$. Then a must also divide bc. But this contradicts our premise that $a \nmid bc$! Therefore a must not actually divide b.

13) Use a direct proof to show that the sum of two odd integers is even.
If m is odd, then it is $2k+1$ for some integer k. Therefore, the sum of **two** odd integers would be $(2k+1)+(2j+1) = 2k+2j+2 = 2(k+j+1)$. Since this result is clearly even, we have our proof.

14) Prove this identity law for sets: $A \cap U = A$. [Hint: What does it mean for two sets to be equal?]

Let x be an element of $A \cap U$. Then $x \in A$ and $x \in U$. Thus $x \in A$ by simplification. And so every element of $A \cap U$ is also an element of A.

Going the other direction, let x be an element of A. But, since x is a member of the universe of discourse, x is also an element of U. Thus $x \in A$ and $x \in U$ by conjunction. Thus, by definition, $x \in A \cap U$. Since every element of A is also in $A \cap U$ and vice versa, these two sets are subsets of one another. And, therefore, these two sets must be equal. □

15) Letting $A = \{a, b, d\}$ and $B = \{a, b, c, e, f\}$, find the following:

i) $A \cap B$
$\{a, b\}$

ii) $A \cup B$
$\{a, b, c, d, e, f\}$

iii) $A - B$
$\{d\}$

iv) $B - A$
$\{c, e, f\}$

16) Prove by induction:

$$\sum_{k=0}^{n} \left(\frac{-1}{2}\right)^k = \frac{2}{3} + \frac{1}{3} \cdot \left(\frac{-1}{2}\right)^n$$

for all natural numbers n. (That is, $n \geqslant 0$.)

The base case is nearly trivial. When $n = 0$, the left side is the sum of a single 1: 1. The right side, similarly, is merely two-thirds plus one-third (times 1): 1.

The inductive hypothesis is that for some natural number $q \geqslant 0$, the proposition holds true. That is:

$$\sum_{k=0}^{q} \left(\frac{-1}{2}\right)^k = \frac{2}{3} + \frac{1}{3} \cdot \left(\frac{-1}{2}\right)^q$$

And we must show that the proposition holds true for $q + 1$ or:

$$\sum_{k=0}^{q+1} \left(\frac{-1}{2}\right)^k = \frac{2}{3} + \frac{1}{3} \cdot \left(\frac{-1}{2}\right)^{q+1}$$

Starting from the left side as usual:

$$\begin{aligned}
\sum_{k=0}^{q+1} \left(\tfrac{-1}{2}\right)^k &= \sum_{k=0}^{q} \left(\tfrac{-1}{2}\right)^k + \left(\tfrac{-1}{2}\right)^{q+1} \\
&= \tfrac{2}{3} + \tfrac{1}{3} \cdot \left(\tfrac{-1}{2}\right)^q + \left(\tfrac{-1}{2}\right)^{q+1} \\
&= \tfrac{2}{3} + \left(\tfrac{1}{3} \div \tfrac{-1}{2} + 1\right) \left(\tfrac{-1}{2}\right)^{q+1} \\
&= \tfrac{2}{3} + \left(\tfrac{-2}{3} + 1\right) \left(\tfrac{-1}{2}\right)^{q+1} \\
&= \tfrac{2}{3} + \tfrac{1}{3} \left(\tfrac{-1}{2}\right)^{q+1}
\end{aligned}$$

Ta-da!

17) You are given this fact: "If $n > 2$ is a prime, then n is odd." Then you are asked to prove this proposition: "If p is a prime with $p > 2$, then $p + 1$ is not prime." [Note that this proposition is *for all* primes greater than 2.]

Using the choose method, we pick a representative prime from the set involved — those greater than 2 — call it k. This prime must be odd by the original fact given. Since k is odd, it must be representable as $2m + 1$ for some integer m. If we add one to k, that must be equal to $2m+1+1 = 2m+2 = 2(m + 1)$. Since this is clearly even, it is divisible by 2 — but not 2 — and so is not prime. □

Bonus Problems:

18) State what the following statement means when the *or* is inclusive vs. when it is exclusive. Which type of *or* do you think was intended?

Dinner for two includes two items from column A or three items from column B.

With an inclusive *or*, the statement means that either two column-A items or three column-B items or both are included with the two-person dinner.

With an exclusive *or*, the statement means that either two column-A items or three column-B items but not both are included with the two-person dinner.

I believe they meant exclusive *or* here. The restaurant won't get rich selling five items for the same price as two or three.

19) Verify this distributive law of Boolean algebra: $x \cdot (y+z) \equiv (x \cdot y) + (x \cdot z)$.

x	y	z	$\frac{A}{y+z}$	$\frac{B}{x \cdot A}$	$\frac{C}{x \cdot y}$	$\frac{D}{x \cdot z}$	$\frac{E}{C+D}$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

We note that all entries in columns B and E are identical.

Now verify this distributive law (also of Boolean algebra): $x + (y \cdot z) \equiv (x+y) \cdot (x+z)$.

This is immediately true by the duality principle. After all, we've merely changed $+$ to $\cdot$ and vice versa.

20) Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers.

i) $\exists x \forall y (xy = y)$

There is a multiplicative identity for the real numbers such that any real number multiplied by this quantity produces the original number identically.

ii) $\forall x \forall y \exists z (x + y = z)$

The real numbers are closed under addition. That is, adding any two real numbers results in a third real number.

21) The term "relatively prime" means ____.

A) a number has only one or two more factors beyond itself and 1
B) a pair of values share no common factors other than 1
C) he was my first cousin until the divorce
D) one of the pair of values must be prime
E) primality is a relative thing and must be gauged in context

22) The Well-Ordering Principle only applies ____.

A) in conjunction with an existential quantifier
B) to non-empty sets
C) to sets of integers
D) to sets of natural numbers
E) when using induction

23) What is the power set of $\{a, b, c\}$?
$\{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

24) Show that $(p \to q) \to r$ and $p \to (q \to r)$ are not equivalent. (And thus that implication is not associative.)

This can be done with a truth table:

p	q	r	A: $p \to q$	B: $A \to r$	C: $q \to r$	D: $p \to C$
true	true	true	true	true	true	true
true	true	false	true	false	false	false
true	false	true	false	true	true	true
true	false	false	false	true	true	true
false	true	true	true	true	true	true
false	true	false	true	**false**	false	**true**
false	false	true	true	true	true	true
false	false	false	true	**false**	true	**true**

Note how the last and second-to-last entries differ for the propositions under consideration (columns B and D).

25) Prove the given fact from #17: "If $n > 2$ is prime, then n is odd." [Hint: The premise could have also been written as "$n > 2$ and n is prime".] [Hint 2: An indirect approach might be easiest.]

An indirect approach will require the contrapositive implication: "If n is even, then either $n \leqslant 2$ or n is not prime." If n is even, then we can write it as $2k$ for some integer k. Let's proceed by cases on k: $k \leqslant 1$ and $k > 1$.

Case 1: If $k \leqslant 1$, then $2k \leqslant 2$ and thus the left clause is true.

Case 2: If $k > 1$, then $2k > 2$. But does this mean n is not prime? Yes, because it is not 2 and yet has a factor of 2 — therefore a factor not itself or 1.

Having proved both clauses, we can use the principle of addition to combine them and, since the original implication is logically equivalent to the contrapositive, we've proven the entire 'fact'. □

26) Express the following English sentence using predicates, quantifiers, and logical operations.

> Each participant on the conference call whom the host of the call did not put on a special list was billed.

Domain: all callers.

$C(x) = x$ participated in the conference call

$L(x) = x$ was listed specially by the conference call's host

$B(x) = x$ was billed for the conference call

$\forall x((C(x) \wedge \neg L(x)) \to B(x))$

What is the opposite of the sentence? (It may help to negate your symbolic version first.)

The opposite of $\forall x((C(x) \wedge \neg L(x)) \to B(x))$ would be:

$$\begin{aligned} \neg(\forall x((C(x) \wedge \neg L(x)) \to B(x))) &= \exists x \neg((C(x) \wedge \neg L(x)) \to B(x)) \\ &= \exists x \neg(\neg(C(x) \wedge \neg L(x)) \vee B(x)) \\ &= \exists x(C(x) \wedge \neg L(x) \wedge \neg B(x)) \end{aligned}$$

Therefore we have in English something like, "There exists a caller who was not billed for the conference call and who both did participate in the conference call and was not listed specially by the conference call's host." Since that is a bit wordy, we may wish to try harder to compact it. Perhaps, "Someone wasn't billed for the conference call and yet they were in on the call and were not given special permission by the call's host."