Our goal here is to solve the Fibonacci sequence into closed form again but this time with generating functions!
Let's start with the recurrence relation for the sequence:

$$\begin{aligned} f_0 &= 1 \\ f_1 &= 1 \\ f_n &= f_{n-1} + f_{n-2}, n > 1 \end{aligned}$$

Now, for our generating function. Our goal is a generating function that holds all the Fibonacci numbers as coefficients:

$$F(z) = \sum_{n \geqslant 0} f_n z^n = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + 13z^6 + \dots$$

So let's start by multiplying our recurrence by our z^n powers:

$$f_n z^n = f_{n-1} z^n + f_{n-2} z^n, n > 1$$

And now sum up those equations over all valid n values:

$$\sum_{n>1} f_n z^n = \sum_{n>1} f_{n-1} z^n + \sum_{n>1} f_{n-2} z^n$$

Now, let's make the left side look like $F(z)$ by including the missing terms. This requires us to add them into the sum and remove them outside the sum to maintain equality with what was there before!

$$-1 - 1z + \sum_{n \geqslant 0} f_n z^n = \sum_{n>1} f_{n-1} z^n + \sum_{n>1} f_{n-2} z^n$$

The other two will require a little shifting to line up with the powers and subscripts. Let's factor out some zs:

$$F(z) - z - 1 = z \sum_{n>1} f_{n-1} z^{n-1} + z^2 \sum_{n>1} f_{n-2} z^{n-2}$$

Now we can substitute $k = n - 1$ and $m = n - 2$ to simplify the right-side summations:

$$F(z) - z - 1 = z \sum_{k>0} f_k z^k + z^2 \sum_{m \geqslant 0} f_m z^m$$

Now the m summation is ready to be $F(z)$ and the k one is just a single term off, so:

$$F(z) - z - 1 = z(F(z) - 1) + z^2 F(z)$$

Next we collect the $F(z)$ terms to the left side and everything else to the right:

$$\begin{aligned} F(z) - zF(z) - z^2 F(z) &= z - z + 1 \\ F(z)(1 - z - z^2) &= 1 \\ F(z) &= \frac{1}{1 - z - z^2} \end{aligned}$$

And this should be it, right? But that's just the generating function! We don't have the closed form, yet.
We have a crazy denominator here. It looks like:

$$1 - z - z^2 = (1 + az)(1 + bz)$$

But for what a and b? Well, it turns out that they are:

$$\frac{-1 \pm \sqrt{5}}{2}$$

So, for partial fraction decomposition, we have:

$$\begin{aligned}
\frac{1}{1-z-z^2} &= \frac{A}{1+az} + \frac{B}{1+bz} \qquad = A\sum_{n \geqslant 0}(-a)^n z^n + B\sum_{n \geqslant 0}(-b)^n z^n \\
&= \frac{A(1+bz)+B(1+az)}{(1+az)(1+bz)}
\end{aligned}$$

Now, equating the numerators:

$$\begin{aligned}
1 &= A + Abz + B + Baz \\
&= A + B + (Ab + Ba)z
\end{aligned}$$

So, it would seem that $A + B = 1$ so we know, for instance, that $B = 1 - A$. Also:

$$\begin{aligned}
0 &= Ab + Ba \\
0 &= Ab + (1-A)a \\
-a &= A(b-a) \\
A &= \frac{a}{a-b} \\
&= \frac{5-\sqrt{5}}{10}
\end{aligned}$$

So, B must be:

$$\begin{aligned}
B &= 1 - \frac{a}{a-b} \\
&= \frac{a-b-a}{a-b} \\
&= \frac{b}{b-a} \\
&= \frac{5+\sqrt{5}}{10}
\end{aligned}$$

Thus our solutions are:

$$A(-a)^n + B(-b)^n = \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

Just like we got before, but with generating functions! Yea!!!